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NONLINEAR ELECTROHYDRODYNAMIC STABILITY OF A POISEUILLE TWO-LAYER FLOW

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The long-wave stability of the Poiseuille two-layer flow of homogeneous viscous dielectrics between plate electrodes under a constant potential difference is studied in an electrohydrodynamic approximation. A linear asymptotic stability analysis shows that surface polarization forces are a destabilizing factor, in addition to viscous stratification. The method of many scales is used to obtain the Kuramoto–Sivashinsky equation governing the weakly nonlinear evolution of the interface between the dielectrics. Within the framework of the approaches used, it is shown that nonlinear interactions limit perturbation growth and the interface does not fail even for a rather large potential difference.

As is known, surface polarization forces are a factor responsible for instability of the plane interface between two quiescent viscous dielectric fluids in a normal homogeneous electric field [1]. A linear analysis of the long-wave stability shows that these forces also exert a destabilizing effect in the case where the basic state of the system is a Poiseuille two-layer flow [2]. In the present work, the long-wave electrohydrodynamic stability of the Poiseuille two-layer flow of dielectric fluids between electrodes under a constant potential difference is studied in a weakly nonlinear approximation. A similar problem without an electric field was studied in a linear approximation [3] and in a weakly nonlinear approximation [4–6]. The methods used to study the long-wave stability were proposed in [3, 4].

In a previous paper [2], we considered a possible scenario of the nonlinear evolution of perturbations without using a linear stability analysis because there is no exact correspondence between the linear dispersion relations of the complete original problem and the reduced weakly nonlinear model.

1. We consider the Poiseuille two-layer flow of immiscible dielectric fluids between plate horizontal electrodes under constant potential difference Φ^*. We assume that extraneous charges are absent at the interface and in the fluid volumes, the electrodes are ideal conductors, and both fluids are viscous, incompressible, homogeneous, isotropic dielectrics having the same temperature and density. In addition, the absence of a gravitational field is assumed without loss of generality in the formulation of the problem.

We restrict ourselves to the two-dimensional case. During motion, the fluids occupy the regions $\Omega_1 = \{-d < y < H(x,t), -\infty < x < \infty\}$ and $\Omega_2 = \{H(x,t) < y < d, -\infty < x < \infty\}$, where x and y are rectangular Cartesian coordinates (the y axis is directed perpendicular to the planes of the electrodes), t is time, and d is a positive constant (see Fig. 1). The coefficients 1 and 2 denote the quantities that correspond to the regions Ω_1 and Ω_2, respectively. Let U_j and V_j be the x and y velocity-vector components, P_j the pressure, and Φ_j the electric-field potential in the region Ω_j (here and below, $j = 1$ and 2).

The flow is characterized by the following constant physical parameters: ρ is the density, μ_j are the dynamic viscosities, ε_j are the dielectric constants of the fluids, and σ is the interfacial tension coefficient.

It is assumed that there is no external magnetic field. The physical system considered is described

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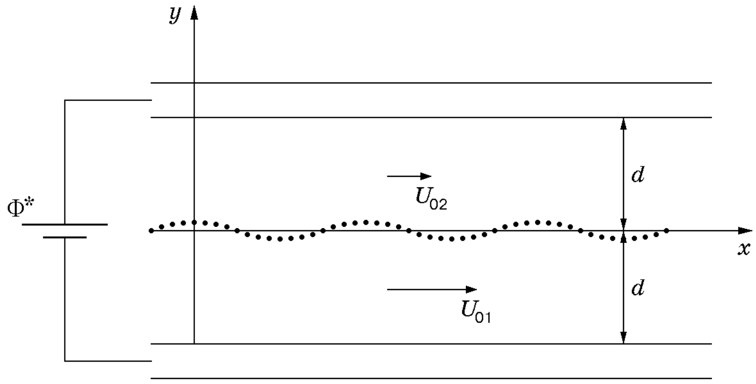

Fig. 1

using the electrohydrodynamic approximation of [7]. Within the framework of the above assumptions, the governing equations and the boundary conditions have the form

$$\rho(U_{jt} + U_j U_{jx} + V_j U_{jy}) = -P_{jx} + \mu_j(U_{jxx} + U_{jyy}),$$

$$\rho(V_{jt} + U_j V_{jx} + V_j V_{jy}) = -P_{jy} + \mu_j(V_{jxx} + V_{jyy}), \tag{1.1}$$

$$U_{jx} + V_{jy} = 0, \qquad \Phi_{jxx} + \Phi_{jyy} = 0;$$

$$\begin{aligned} &\Phi_1 = 0, \quad U_1 = 0, \quad V_1 = 0 \quad \text{for} \quad y = -d, \\ &\Phi_2 = \Phi^*, \quad U_2 = 0, \quad V_2 = 0 \quad \text{for} \quad y = d. \end{aligned} \tag{1.2}$$

At the interface $y = H(x,t)$, we impose the continuity conditions for the electric induction vector component normal to the interface, the potential, the velocity vector, and the normal and shearing stresses and the kinematic nonpenetration condition:

$$[\varepsilon(\Phi_y - H_x\Phi_x)] = 0, \quad [\Phi] = 0, \quad [U] = 0, \quad [V] = 0,$$

$$\begin{aligned} &-[P] + 2(1 + H_x^2)^{-1}[\mu(V_y - H_x(U_y + V_x) + H_x^2 U_x)] \\ &+ (1 + H_x^2)^{-1}[(8\pi)^{-1}\varepsilon((1 - H_x^2)(\Phi_y^2 - \Phi_x^2) - 4H_x\Phi_x\Phi_y)] = \sigma H_{xx}(1 + H_x^2)^{-3/2}, \end{aligned} \tag{1.3}$$

$$[\mu(2H_x(V_y - U_x) + (1 - H_x^2)(U_y + V_x))] = 0, \qquad H_t + U_1 H_x = V_1.$$

Here $[(\,\cdot\,)] \equiv (\,\cdot\,)_1 - (\,\cdot\,)_2$ is the jump of the quantity at the interface; the subscripts denote partial derivatives.

2. We consider the problem of the stability of the stationary plane–parallel flow induced by a constant pressure gradient $-F$ ($F > 0$) along the x axis. The flow is described by the following solution of problem (1.1)–(1.3):

$$U_{0j} = -\frac{F}{2\mu_j}\,y^2 - \frac{Fd(m-1)}{2\mu_j(m+1)}\,y + \frac{Fd^2}{\mu_1 + \mu_2}, \quad V_{0j} = 0, \quad P_{0j} = -Fx + P_j',$$

$$\Phi_{01} = \Phi^*\,\frac{\varepsilon_2(y+d)}{d(\varepsilon_1 + \varepsilon_2)}, \qquad \Phi_{02} = \Phi^*\,\frac{\varepsilon_1(y + \gamma d)}{d(\varepsilon_1 + \varepsilon_2)}, \qquad H_0 = 0,$$

where $m = \mu_2/\mu_1$, $\gamma = \varepsilon_2/\varepsilon_1$, and $P_j' = \text{const}$.

For simplicity, we restrict ourselves to the case with identical thicknesses of the layers in the unperturbed state. Let $U_j - U_{0j} = Uu_j$, $V_j = Uv_j$, $P_j - P_{0j} = \rho U^2 p_j$, $\Phi_j - \Phi_{0j} = \Phi^*\varphi_j$, $H = dh$, $x \to dx$, $y \to dy$, and $t \to dU^{-1}t$, where $U = U_0(0) = Fd^2/(\mu_1 + \mu_2)$ is the unperturbed flow velocity at the plane interface.

The functions ψ_j are given by the relations $u_j = \psi_{jy}$ and $v_j = -\psi_{jx}$. The evolution of the perturbations is described by the system

$$\mathrm{Re}_j(\nabla^2\psi_{jt} + (u_{0j} + \psi_y)\nabla^2\psi_{jx} - \psi_x(u_{0yy} + \nabla^2\psi_{jy})) = \nabla^4\psi_j, \quad \nabla^2\varphi_j = 0, \tag{2.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\varphi_1 = 0, \quad \psi_{1y} = 0, \quad \psi_{1x} = 0 \quad \text{for} \quad y = -1,$$

$$\varphi_2 = 0, \quad \psi_{2y} = 0, \quad \psi_{2x} = 0 \quad \text{for} \quad y = 1,$$

$$[u_0 + \psi_y] = 0, \qquad [\psi_x] = 0, \qquad [\varphi_0 + \varphi] = 0,$$

$$\mathrm{Re}_1[\psi_{yt} + u_0\psi_{yx} + \psi_y\psi_{yx} - u_{0y}\psi_x - \psi_x\psi_{yy}] - [m'(\psi_{yxx} + \psi_{yyy})] + f_x = 0, \tag{2.2}$$

$$f = 2(1 + h_x^2)^{-1}[m'((h_x^2 - 1)\psi_{yx} - h_x(u_{0y} + \psi_{yy} - \psi_{xx}))]$$

$$+ (1/2)(1 + h_x^2)^{-1}[\gamma((1 - h_x^2)(\varphi_{0y}^2 + 2\varphi_{0y}\varphi_y + \varphi_y^2 - \varphi_x^2) - 4h_x\varphi_x(\varphi_{0y} + \varphi_y))] - \mathrm{We}\, h_{xx}(1 + h_x^2)^{-3/2},$$

$$[m'(-4h_x\psi_{yx} + (1 - h_x^2)(u_{0y} + \psi_{yy} - \psi_{xx}))] = 0, \quad [\gamma(\varphi_{0y} + \varphi_y - h_x\varphi_x)] = 0,$$

$$h_t + (u_{01} + \psi_{1y})h_x = -\psi_{1x} \quad \text{for} \quad y = h(x, t).$$

Here $\varphi_{01} = \gamma(y+1)/(1+\gamma)$, $\varphi_{02} = (y+\gamma)/(1+\gamma)$, $u_{0j} = b_j y^2 + a_j y + 1$, $a_1 = (1-m)/2$, $a_2 = (1-m)/(2m)$, $b_1 = -(m+1)/2$, $b_2 = -(m+1)/(2m)$, $\mathrm{Re}_j = \rho U d/\mu_j$, $\mathrm{We} = \sigma/(\mu_1 U)$, $m'_j = \mu_j/\mu_1$, and $\gamma_j = \varepsilon_j \Phi^{*2}/(4\pi d\mu_1 U)$.

3. The time evolution of small long-wave perturbations of the type of normal modes

$$(\psi, \varphi, h) = (\psi(y), \varphi(y), h)\exp(i\alpha(x - ct)), \tag{3.1}$$

where $\alpha \in \mathbb{R}$ is the wave number ($\alpha \ll 1$) and $c \in \mathbb{C}$, is determined by solving a spectral problem with a regularly perturbed parameter α that is obtained by linearization of the equations and conditions (2.1) and (2.2) for the basic state specified above and substitution of relations (3.1) into the system obtained.

Let us impose the following constraints on the orders of magnitude of the determining parameters:

$$\mathrm{Re}_j = O(1), \qquad \mathrm{We} = O(\alpha^{-2}), \qquad \gamma_j = O(1). \tag{3.2}$$

In addition, the viscosities of the fluids are assumed to be different ($m \neq 1$). Then, [2]

$$c = c^{(0)} + \alpha c^{(1)} + O(\alpha^2), \tag{3.3}$$

where

$$c^{(0)} = 1 + \frac{2(m-1)^2}{m^2 + 14m + 1},$$

$$c^{(1)} = \frac{2i\,\mathrm{Re}_1\,(m-1)^2}{(m^2 + 14m + 1)^2}H(m) + \frac{i\gamma_2(1-\gamma)^2(m+1)}{3(1+\gamma)^3(m^2 + 14m + 1)} - \frac{i\alpha^2\,\mathrm{We}\,(1+m)}{3(m^2 + 14m + 1)},$$

$$H(m) = -h_1(-1) - h_1'(-1) - 7h_2(1) + 3h_2'(1) - 7mh_1(-1) - 3mh_1'(-1) - mh_2(1) + mh_2'(1),$$

$$h_1 = -\frac{m^2 - 1}{1680}\,y^7 - \frac{(m-1)^2}{480}\,y^6 - \frac{m^4 + 18m^3 - 156m^2 - 98m - 21}{480(m^2 + 14m + 1)}\,y^5 - \frac{m^3 - 17m^2 - 17m + 1}{24(m^2 + 14m + 1)}\,y^4, \tag{3.4}$$

$$h_2 = -\frac{m^2 - 1}{1680m^2}\,y^7 - \frac{(m-1)^2}{480m^2}\,y^6 - \frac{21m^4 + 98m^3 + 156m^2 - 18m - 1}{480m^2(m^2 + 14m + 1)}\,y^5 - \frac{m^3 - 17m^2 - 17m + 1}{24m(m^2 + 14m + 1)}\,y^4.$$

From (3.3) it follows that if the dielectrics of the fluids ($\gamma \neq 1$) are different, the transverse electric field destabilizes the flow considered, and the instability increment is proportional to the square of the wave number. It is also obvious that for rather small values of the parameter We [We $< O(\alpha^{-2})$], which characterizes the interfacial tension, the interface is unstable for any potential difference between the electrodes.

We note that the quantity $H(m)$ given by (3.4) is positive. This corresponds to Yih instability [3], which results from the difference in the velocity gradients of the main flow at the interface ($m \neq 1$). Yih instability occurs for any Reynolds numbers if the effective interfacial tension is rather small. The instability increment is proportional to the quantity α^2.

As is known, under certain conditions, the perturbation growth due to Yih instability is limited at the nonlinear stage of development of the instability [4]. It is expected that a similar phonomena occurs in our case, too. We study the problem of the stability of the flow considered in a weakly nonlinear approximation.

4. Following the approach used in [4, 6], where a similar problem was studied in the case of no electric field, we assume that in a rather small neighborhood of the critical values of the determining parameters, the characteristic scales of perturbation growth correspond to the linear stage of development of instability. According to the above results of asymptotic linear analysis in the range of values of the determining parameters (3.2), the evolution of small long-wave perturbations (3.1) is determined by the parameter $\exp\,(i\alpha(x - c^{(0)}t) - i\alpha^2 c^{(1)}t)$, where $\alpha \ll 1$. According to this, we set

$$\xi = \varepsilon(x - c^{(0)}t), \qquad \tau = \varepsilon^2 t, \qquad h = \varepsilon A(\xi, \tau),$$

$$\psi_j(x, y, t) = \varepsilon\psi_j^{(0)}(\xi, y, \tau) + \varepsilon^2\psi_j^{(1)}(\xi, y, \tau) + \ldots, \tag{4.1}$$

$$\varphi_j(x, y, t) = \varepsilon\varphi_j^{(0)}(\xi, y, \tau) + \varepsilon^2\varphi_j^{(1)}(\xi, y, \tau) + \ldots;$$

$$\mathrm{Re}_j = O(1), \qquad \mathrm{We} = O(\varepsilon^{-2}), \qquad \gamma_j = O(1), \tag{4.2}$$

where $\varepsilon > 0$ is a small parameter.

Substitution of (4.1) into (2.1) and (2.2) gives a sequence of problems in different approximations for the small parameter ε. The solution of the problem in the approximation $O(\varepsilon)$ is given by

$$\varphi_1^{(0)}(\xi, y, \tau) = A(\xi, \tau)\,\frac{\gamma(\gamma - 1)(y + 1)}{(1 + \gamma)^2}, \qquad \varphi_2^{(0)}(\xi, y, \tau) = A(\xi, \tau)\,\frac{(\gamma - 1)(y - 1)}{(1 + \gamma)^2},$$

$$\psi_j^{(0)}(\xi, y, \tau) = A(\xi, \tau)(c^{(0)} - 1)(1 + B_j y + C_j y^2 + D_j y^3),$$

where $B_1 = (7 + m)/4$, $B_2 = (-1 - 7m)/(4m)$, $C_1 = (1 + m)/2$, $C_2 = (1 + m)/(2m)$, $D_1 = (-1 + m)/4$, and $D_2 = (-1 + m)/(4m)$.

In the approximation $O(\varepsilon^2)$, the quantities A_ξ, A^2, and $A_{\xi\xi\xi}$ appear in the equations and boundary conditions. Therefore, in this approximation, we seek a solution of the problem in the form $\psi_j^{(1)}(\xi, y, \tau) = A_\xi\varphi_{1j}(y) + A_{\xi\xi\xi}\varphi_{2j}(y) + A^2\varphi_{3j}(y)$. After certain algebraic calculations, from the kinematic condition [the last relation in (2.2)], we obtain the equation for the function $A(\xi, \tau)$, which is a necessary condition for the solvability of the problem and determines the evolution of the interface:

$$A_\tau + 2QAA_\xi + EA_{\xi\xi} + SA_{\xi\xi\xi\xi} = 0. \tag{4.3}$$

Here

$$E = \varphi_{11}(0) = E_1 + E_2, \qquad E_1 = \frac{2\mathrm{Re}_1(m - 1)^2}{(m^2 + 14m + 1)^2}\,H(m) > 0,$$

$$E_2 = \frac{\gamma_2(1 - \gamma)^2(m + 1)}{3(1 + \gamma)^3(m^2 + 14m + 1)} > 0, \qquad S = \varphi_{21}(0) = \frac{\varepsilon^2\,\mathrm{We}\,(1 + m)}{3(m^2 + 14m + 1)} > 0,$$

$$Q = \frac{1}{2}\,(u_{01y}(0) + u_{01yy}) + \varphi_{31}(0) = \frac{(m - 1)(m^2 + 6m + 7)}{4(m^2 + 14m + 1)}.$$

The quantity $H(m)$ is given by (3.4).

The equation obtained differs from that derived in [4] only in the presence of the term $E_2 A_{\xi\xi}$, which is due to electrohydrodynamic effects.

Linearizing Eq. (4.3) for the basic state $A = 0$, we have the dispersion relation [for the harmonic with the factor $\exp(\lambda t + i\alpha x)$] $\lambda = E\alpha^2 - S\alpha^4$, which, as should be expected, corresponds to (3.3).

5. An equation of the form (4.3), known as the Kuramoto–Sivashinsky equation, is the simplest universal model for nonlinear processes in dissipative systems with long-wave instability and is encountered in various problems, for example, in descriptions of liquid film flows and in plasma physics. This equation has been extensively studied both numerically and analytically.

The periodic wave modes observed in numerical calculations $[A(t, 0) = A(t, L)]$, described by Eq. (4.3), are limited (see [8–10]). The stabilization mechanism involves successive energy transfer from long-wave to short-wave modes during nonlinear interaction and energy dissipation due to the work of interfacial tension forces. With increase in the bifurcation parameter $\mu = (L/(2\pi))(E/S)^{1/2}$, different types of ordered limiting regimes alternate with regions of irregular behavior. For rather large values of μ, oscillations modes of a random character are established [8–10], but the solutions are still limited.

Thus, within the framework of the approximations considered, nonlinear interactions limit perturbation growth and the interface does not fail even for rather large potential differences.

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